

# MINIMAL RAYS ON SURFACES OF GENUS GREATER THAN ONE – PART II

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**ABSTRACT.** We consider any Finsler metric on a closed, orientable surface of genus greater than one. H. M. Morse proved that we can associate an asymptotic direction to minimal rays in the universal cover (in the Poincaré disc: a point on the unit circle). We prove here that, if two minimal rays have a common asymptotic direction, which is not a fixed point of the group of deck transformations, then the two rays can intersect at most in a common initial point. This has strong consequences for the structure of the set of minimal geodesics, as well as for the set of Busemann functions associated to the Finsler metric.

## 1. INTRODUCTION AND MAIN RESULTS

Let us introduce the relevant objects. Let  $M$  be a closed, orientable surface of genus  $> 1$  and  $X$  its universal cover. On  $M$ , there exists a (hyperbolic) Riemannian metric  $g$  of constant curvature  $-1$ , which can be lifted to  $X$ , making  $(X, g)$  isometric to the Poincaré disc model of the hyperbolic plane, i.e.

$$X = \{z \in \mathbb{C} : |z| < 1\}, \quad g_z(v, w) = 4 \cdot (1 - |z|^2)^{-2} \langle v, w \rangle,$$

writing  $\langle \cdot, \cdot \rangle, |\cdot|$  for the euclidean inner product and its norm in  $\mathbb{C}$ . The distance induced by  $g$  on  $X$  will be denoted by  $d_g$ . We write  $\Gamma \subset \text{Iso}(X, g)$  for the group of deck transformations  $\tau : X \rightarrow X$  with respect to the covering  $X \rightarrow M = X/\Gamma$ , which extend to naturally to  $S^1$ . Let us write

$$\text{Fix}(\Gamma) = \{\xi \in S^1 \mid \exists \tau \in \Gamma - \{\text{id}\} : \tau\xi = \xi\}.$$

The  $g$ -geodesics  $\gamma$  in  $X$  are circle segments meeting the “boundary at infinity”  $S^1 = \{z \in \mathbb{C} : |z| = 1\}$  orthogonally; the endpoints of  $\gamma$  are denoted by  $\gamma(-\infty), \gamma(\infty) \in S^1$ . Let  $\mathcal{G}$  be the set of all oriented, unparametrized  $g$ -geodesics  $\gamma \subset X$ . Recall that any  $\tau \in \Gamma - \{\text{id}\}$  has exactly two distinct fixed points in  $S^1$ , which are connected by a unique  $g$ -geodesic  $\gamma \in \mathcal{G}$  being invariant under  $\tau$ ,  $\tau\gamma = \gamma$ ; we will call  $\gamma$  the axis of  $\tau$ .

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We consider any Finsler metric  $F : TX \rightarrow \mathbb{R}$ , which is assumed to be invariant under  $\Gamma$  (i.e. being the lift of a Finsler metric on  $M$ ). We write  $SX = \{F = 1\} \subset TX$  and  $c_v : \mathbb{R} \rightarrow X$  for the geodesic with respect to  $F$  defined by  $\dot{c}_v(0) = v$ . We are interested in *rays* and *minimal geodesics* (with respect to  $F$ ), that is  $F$ -geodesics  $c : [0, \infty) \rightarrow X$ ,  $c : \mathbb{R} \rightarrow X$ , respectively that, in the universal cover  $X$  of  $M$ , minimize the  $F$ -length between any of their points. Often, the first step towards understanding the geodesic flow of  $F$  is to study the minimal geodesics – their behavior is to some extent prescribed by the topology of the underlying manifold and hence minimal geodesics form a basic framework for all geodesics; moreover, since the geodesic flow is defined in terms of local minimizers a variational problem, it is natural to first study the global minimizers of the problem.

In 1924, H. M. Morse [Mor24] studied the global behavior of minimal geodesics on  $M$  and proved two theorems, which we recall now.

**Theorem 1.1** (Morse [Mor24]). *In the notation above, rays and minimal geodesics in  $(X, F)$  tend to  $\pm\infty$  in tubes around  $g$ -geodesics  $\gamma \in \mathcal{G}$  of finite width  $D$ , where  $D$  is a constant depending only on  $F$  and  $g$ . In particular, rays  $c : [0, \infty) \rightarrow X$  have a well-defined endpoint  $c(\infty) \in S^1$ .*

For  $\xi \in S^1$  and  $\gamma \in \mathcal{G}$  set

$$\mathcal{R}_+(\xi) := \{v \in SX \mid c_v : [0, \infty) \rightarrow X \text{ is a ray, } c(\infty) = \xi\},$$

$$\mathcal{M}(\gamma) := \{v \in SX \mid c_v : \mathbb{R} \rightarrow X \text{ is a minimal geodesic, } c(\pm\infty) = \gamma(\pm\infty)\}.$$

Hence, any  $F$ -ray in  $X$  belongs to some  $\mathcal{R}_+(\xi)$ , while any  $F$ -minimal geodesic belongs to some  $\mathcal{M}(\gamma)$ . Conversely, we have  $\mathcal{M}(\gamma) \neq \emptyset$  for all  $\gamma \in \mathcal{G}$  and  $\mathcal{R}_+(\xi)$  projects to all of  $X$  for all  $\xi \in S^1$ .

The second theorem clarifies the structure of some of the sets  $\mathcal{R}_+(\xi)$  for fixed  $\xi \in S^1$ . If  $\tau \in \Gamma - \{\text{id}\}$  has as its axis some  $\gamma \in \mathcal{G}$  and if  $\xi = \gamma(\infty) \in \text{Fix}(\Gamma)$  is the corresponding fixed point of  $\tau$ , then the structure of  $\mathcal{R}_+(\xi)$  can be described as follows, cf. Figure 1. Let us write

$$\mathcal{M}_{\text{per}}(\gamma) := \{v \in \mathcal{M}(\gamma) : \tau c_v(\mathbb{R}) = c_v(\mathbb{R})\}.$$

**Theorem 1.2** (Morse [Mor24]). *If  $\tau \in \Gamma$  and if  $\gamma \in \mathcal{G}$  is the axis of  $\tau$  with endpoint  $\xi := \gamma(\infty) \in \text{Fix}(\Gamma)$ , then  $\mathcal{M}_{\text{per}}(\gamma) \neq \emptyset$ . Moreover, no ray from  $\mathcal{R}_+(\xi) - \mathcal{M}_{\text{per}}(\gamma)$  can intersect any minimal geodesic from  $\mathcal{M}_{\text{per}}(\gamma)$  and every ray in  $\mathcal{R}_+(\xi)$  is asymptotic near  $+\infty$  to some minimal geodesic from  $\mathcal{M}_{\text{per}}(\gamma)$ . All minimal geodesics in  $\mathcal{M}(\gamma) - \mathcal{M}_{\text{per}}(\gamma)$  are heteroclinic between a pair of neighboring periodic minimal geodesics in  $\mathcal{M}_{\text{per}}(\gamma)$ .*

Note that in the last statement of Theorem 1.2, we implicitly exclude any minimal geodesic in  $\mathcal{M}(\gamma)$  to be homoclinic to a single periodic minimal geodesic in  $\mathcal{M}_{\text{per}}(\gamma)$ . In particular, if  $\mathcal{M}_{\text{per}}(\gamma)$  consists of only one geodesic, then  $\mathcal{M}(\gamma) = \mathcal{M}_{\text{per}}(\gamma)$ .

Let us remark that in 1932, G. A. Hedlund [Hed32] proved results analogous to Theorems 1.1 and 1.2 for the case of a genus 1 surface, i.e. the

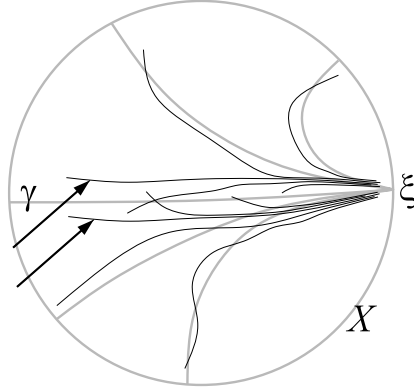


FIGURE 1. The structure of  $\mathcal{R}_+(\xi)$  in  $X \subset \mathbb{C}$  in Theorem 1.2 for  $\xi \in \text{Fix}(\Gamma) \subset S^1$ . Minimal rays with respect to  $F$  are depicted in black, their corresponding  $g$ -geodesics in gray.  $\gamma$  is the axis of  $\tau$  and the two arrows show a pair of  $\tau$ -invariant minimal geodesics in  $\mathcal{M}_{\text{per}}(\gamma)$ .

2-torus  $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$ . Here the minimal geodesics and rays move along straight euclidean lines in the universal cover  $X = \mathbb{R}^2$  and hence also here we can define the sets  $\mathcal{R}_+(\xi)$  with  $\xi \in S^1$ ,  $c(\infty) = \xi$  being the direction of the accompanying line. Then for  $\xi \in S^1$  with rational slope – that is “fixed points” of  $\mathbb{Z}^2$  – the set  $\mathcal{R}_+(\xi)$  has the analogous structure as in the case of genus  $> 1$  in Theorem 1.2, the difference being that, as a set,  $\mathcal{M}_{\text{per}}(\gamma)$  is  $\mathbb{Z}^2$ -invariant, while for genus  $> 1$  it is only  $\tau$ -invariant.

Later, in 1988, V. Bangert [Ban88] proved moreover, that the sets  $\mathcal{R}_+(\xi)$  with  $\xi \in S^1$  of irrational slope have a rather simple structure: the rays in  $\mathcal{R}_+(\xi)$  can intersect at most in common initial points and the set of minimal geodesic with a fixed irrational slope contains no intersecting minimal geodesics in  $\mathbb{R}^2$  (in fact, not even in  $\mathbb{T}^2$ ).

The results of Hedlund and Bangert together give a complete picture of the structure of all rays in the case of the torus. Returning to the case of genus  $> 1$ , what is left open in understanding the structure of all rays is the structure of the sets  $\mathcal{R}_+(\xi)$  with  $\xi \notin \text{Fix}(\Gamma)$ . Building on our work from [Sch14b], we will prove here the following theorem.

**Main Theorem 1.3.** *If  $\xi \in S^1 - \text{Fix}(\Gamma)$ , then*

$$\liminf_{t \rightarrow \infty} d_g(c_v(\mathbb{R}), c_w(t)) = 0 \quad \forall v, w \in \mathcal{R}_+(\xi).$$

Hence, any two rays in  $\mathcal{R}_+(\xi)$  with  $\xi \notin \text{Fix}(\Gamma)$  will approach each other at some points near  $+\infty$ .

**Remark 1.4.** It was previously known [CS14], that if  $F$  is a Riemannian metric on  $M$  with non-positive curvature  $K \leq 0$ , then any flat strip in  $X$ ,

i.e. an isometrically embedded euclidean strip

$$(\mathbb{R} \times [0, a], \langle \cdot, \cdot \rangle_{\text{euc}}) \hookrightarrow (X, F), \quad a > 0,$$

is periodic, i.e. bounded by two axes of some non-trivial  $\tau \in \Gamma - \{\text{id}\}$ . All other flat strips in such a non-positively curved surface have to be “thin” in the ends near  $\pm\infty$ . Main Theorem 1.3 is a generalization of this fact to general Finsler metrics on  $M$ .

Main Theorem 1.3 has a series of corollaries.

**Corollary 1.5.** *If  $\xi \in S^1 - \text{Fix}(\Gamma)$  and if  $v, w \in \mathcal{R}_+(\xi)$  with  $c_w(0) = c_v(a)$  for some  $a > 0$ , then  $c_w(t) = c_v(t + a)$ .*

Hence two rays with a common asymptotic direction not in  $\text{Fix}(\Gamma)$  can intersect at most in a common initial point, which is analogous to V. Bangert’s result in the case of the 2-torus.

One of our original motivations to prove Corollary 1.5 was to obtain the following theorem. We write  $\text{pr}(\mathcal{M})$  for the projection of the set  $\mathcal{M} = \bigcup \{\mathcal{M}(\gamma) : \gamma \in \mathcal{G}\}$  of all minimal geodesics into the  $F$ -unit tangent bundle of  $M$ ,  $B_F(x, r) \subset X$  is the  $F$ -ball of radius  $r$ , centered at any  $x$  in the universal cover  $X$ . Let us denote the geodesic flow of  $F$  by  $\phi_F^t : SX \rightarrow SX$ .

**Theorem 1.6** (cf. [GKOS14]). *If  $M$  is a closed surface and  $g$  is any Riemannian metric on  $M$ , then*

$$h_{\text{top}}(\phi_g^t|_{\text{pr}(\mathcal{M})}) = h(g) := \lim_{r \rightarrow \infty} \frac{1}{r} \log \text{vol}_g B_g(x, r).$$

Here,  $h_{\text{top}}(\phi_g^t|_{\text{pr}(\mathcal{M})})$  denotes the topological entropy of the restricted geodesic flow  $\phi_g^t : \text{pr}(\mathcal{M}) \rightarrow \text{pr}(\mathcal{M})$ ;  $h(g)$  is often called volume growth of  $(M, g)$ . Note that, for general closed Riemannian manifolds, one has the following estimate, cf. Theorem 9.6.7 in [KH96]:

$$h_{\text{top}}(\phi_g^t|_{\text{pr}(\mathcal{M})}) \geq h(g).$$

Although Theorem 1.6 was proved by slightly different means, one can derive the following strengthening of Theorem 1.6, using Corollary 1.5.

**Corollary 1.7.** *For all  $\gamma \in \mathcal{G}$ , the topological entropy of  $\phi_F^t : \mathcal{M}(\gamma) \rightarrow \mathcal{M}(\gamma)$  vanishes, i.e.  $h_{\text{top}}(\phi_F^t|_{\mathcal{M}(\gamma)}) = 0$ .*

To describe the structure of  $\mathcal{R}_+(\xi)$  further, write

$$d_F(x, y) := \inf \left\{ \int_0^1 F(\dot{c}) dt \mid c : [0, 1] \rightarrow X \text{ } C^1, c(0) = x, c(1) = y \right\}$$

for the  $F$ -distance between  $x, y \in X$ . For a sequence  $\{x_n\} \subset X$  and  $\xi \in S^1$  we shall write  $x_n \rightarrow \xi$ , if this is true in the euclidean metric in  $\mathbb{C} \supset X \cup S^1$ . Fixing any “origin”  $o \in X$ , the set of *horofunctions*  $\mathcal{H}_+(\xi)$  of *direction*  $\xi$  is the set of all possible  $C_{\text{loc}}^0$  limit functions  $u : X \rightarrow \mathbb{R}$  of sequences

$$u_n(x) = d_F(o, x_n) - d_F(x, x_n), \quad \text{where } x_n \rightarrow \xi.$$

**Corollary 1.8.** *If  $\xi \in S^1 - \text{Fix}(\Gamma)$ , then the set  $\mathcal{H}_+(\xi)$  consists of precisely one horofunction. Hence, for all sequences  $x_n \rightarrow \xi$ , the sequence of functions  $d_F(o, x_n) - d_F(\cdot, x_n)$  converges in  $C_{loc}^0$  to the unique function  $u \in \mathcal{H}_+(\xi)$ .*

One can describe the set  $\mathcal{R}_+(\xi)$  in terms of the horofunctions in  $\mathcal{H}_+(\xi)$ . From Corollary 1.8, we obtain the following strengthening of Corollary 1.5.

**Corollary 1.9.** *If  $\xi \in S^1 - \text{Fix}(\Gamma)$ , then for all  $\varepsilon > 0$  the set  $\phi_F^\varepsilon(\mathcal{R}_+(\xi)) \subset SX$  is locally a Lipschitz graph over its projection in  $X$ .*

Next, let us observe that most sets  $\mathcal{M}(\gamma)$  are not very big.

**Corollary 1.10.** *For all  $\xi \in S^1$ , there exists a countable set  $A \subset S^1$ , such that  $\mathcal{M}(\gamma)$  consists of only one minimal geodesic for all  $\gamma \in \mathcal{G}$  connecting a point in  $S^1 - A$  to  $\xi$ .*

Given some direction  $\xi_+ \in X$ , then any point  $x \in X$  determines a unique  $\xi_- \in S^1$  as follows.

**Corollary 1.11.** *If  $x \in X$  and  $\xi \in S^1$ , then there exists a unique  $\gamma \in \mathcal{G}$  with  $\gamma(\infty) = \xi$ , such that  $x$  lies in the closed strip between a pair of (not necessarily distinct) minimal geodesics from  $\mathcal{M}(\gamma)$ .*

We proved that certain behavior of minimal geodesics (in particular: intersecting minimal geodesics) in  $\mathcal{M}(\gamma)$  occurs only for axes  $\gamma \in \mathcal{G}$ . As it turns out, the existence of such behavior is very exceptional. To be more precise, let us recall the following special case of the main result from [Sch14c], writing

$$E := \{f : M \rightarrow \mathbb{R} \mid f \text{ is } C^\infty, f(x) > 0 \forall x \in M\}.$$

**Theorem 1.12** (cf. [Sch14c]). *Given any Finsler metric  $F$  on the surface  $M$ , there exists a residual subset  $\mathcal{O}_F \subset E$  (i.e. a countable intersection of open and dense sets) in the  $C^\infty$  topology, such that for the conformally perturbed Finsler metrics  $f \cdot F$  with  $f \in \mathcal{O}_F$ , in any free homotopy class  $M$  there exists precisely one shortest closed geodesic with respect to  $f \cdot F$ .*

Hence, for the *conformally generic* Finsler metrics, that is Finsler metrics  $f \cdot F$  with  $f \in \mathcal{O}_F$ , the sets  $\mathcal{M}_{per}(\gamma)$  in Theorem 1.2 consist of only one minimal geodesic. We obtain the following corollary.

**Corollary 1.13.** *If  $F$  is conformally generic, then the statements of Main Theorem 1.3 and of Corollaries 1.5, 1.8 and 1.9 are true for all  $\xi \in S^1$ .*

Note, that in the case of the 2-torus, the  $\mathbb{Z}^2$ -invariance of the sets  $\mathcal{M}_{per}(\gamma)$  discussed above leads to a quite different picture for generic Finsler metrics: Here, generic Finsler metrics admit *many* horofunctions and intersecting minimal geodesics for *every* rational direction  $\xi \in S^1$ .

We close the introduction with two remarks.

**Remark 1.14.** Differently from what happens on the torus (recall the examples of Riemannian metrics on  $\mathbb{T}^3$  due to G. A. Hedlund [Hed32]), some

of our techniques work also in higher dimensions. In particular, Theorem 1.1 continues to hold [Kli71]. Here the setting is a closed Riemannian manifold  $(M, g)$ , such that  $g$  has everywhere negative sectional curvatures, and then considering an arbitrary Finsler metric  $F$  on  $M$ . This, however, is a topic for future research.

Another way of generalizing our results is to consider non-compact surfaces. It seems very likely to us that under suitable conditions, our main results and arguments continue to work without much alteration, e.g., for Finsler metrics on complete hyperbolic surfaces of finite volume such as the once-punctured torus.

**Remark 1.15.** Another novelty in this paper, compared to the work of Morse, is the use of Finsler instead of Riemannian metrics. It was observed by E. M. Zauzinsky [Zau62], that the results of Morse carry over to these much more general systems. Moreover, it is known that Finsler metrics can be used to describe the dynamics of arbitrary Tonelli Lagrangian systems in high energy levels, cf. [CIPP98]. Let us remark that also in the torus case, all results carry over to general Finsler metrics (hence Tonelli Lagrangian systems), cf. [Zau62] and [Sch14a].

**Structure of this paper.** This paper continues earlier work [Sch14b], where we began to study the dynamics of minimal rays of a Finsler metric on a closed orientable surface of genus at least two. Hence, we refer to [Sch14b] for some proofs. In Section 2 we recall known facts on rays and minimal geodesics in surfaces of higher genus, including some material on horofunctions. Section 3 is devoted to the proof of Main Theorem 1.3, with Subsection 3.1 containing the proofs of its corollaries.

## 2. BASICS ON MINIMAL GEODESICS IN SURFACES OF HIGHER GENUS

We write  $\pi : TX \rightarrow X$  for the canonical projection,  $0_X$  denotes the zero section and  $T_x X = \pi^{-1}(x)$  the fibers. The norm and distance of  $g$  on  $X$  are denoted by  $|\cdot|_g, d_g$ , respectively. Let us recall the definition of a Finsler metric and refer to [BCS00] for basics on Finsler geometry. The reader unfamiliar with Finsler geometry can without much loss think of a Finsler metric as the norm coming from some Riemannian metric.

**Definition 2.1.** *A function  $F : TX \rightarrow [0, \infty)$  is a Finsler metric on  $X$ , if the following conditions are satisfied:*

- (1) (smoothness)  $F$  is  $C^\infty$  in  $TX - 0_X$ ,
- (2) (positive homogeneity)  $F(\lambda v) = \lambda F(v)$  for all  $v \in TX, \lambda \geq 0$ ,
- (3) (strict convexity) the fiberwise Hessian  $\text{Hess}(F^2|_{T_x X})$  of the square  $F^2$  is positive definite at every vector  $v \in T_x X - \{0\}$ , for all  $x \in X$ .

We will for the rest of the paper fix the Finsler metric  $F$  and assume that  $F$  is  $\Gamma$ -invariant, meaning that the  $g$ -isometries  $\tau \in \Gamma$  are also  $F$ -isometries:

$$F(d\tau(v)) = F(v) \quad \forall v \in TX, \tau \in \Gamma.$$

As a consequence of the compactness of  $M$ , the Finsler metric  $F$  is uniformly equivalent to the norm of  $g$ , i.e. there exists a constant  $c_F > 0$ , such that

$$\frac{1}{c_F} \cdot F \leq |\cdot|_g \leq c_F \cdot F.$$

**Notation 2.2.** *We will often drop the dependence of  $F$  of objects that are defined with respect to  $F$ ; e.g. geodesics will refer to  $F$ -geodesics. We will always denote  $g$ -geodesics by  $\gamma, \gamma_n$  etc., while  $F$ -geodesics will be termed  $c, c_n$  etc..*

We write  $SX = \{v \in TX : F(v) = 1\}$  for the unit tangent bundle of  $F$ . The geodesic flow  $\phi_F^t : SX \rightarrow SX$  of  $F$  is given by  $\phi_F^t v = \dot{c}_v(t)$ , where  $c_v : \mathbb{R} \rightarrow X$  is the  $F$ -geodesic with initial velocity  $\dot{c}_v(0) = v$ . We write

$$l_F(c) = \int_a^b F(\dot{c}) dt$$

for the  $F$ -length of absolutely continuous ( $C^{ac}$ ) curves  $c : [a, b] \rightarrow X$  and

$$d_F(x, y) = \inf \{l_F(c) \mid c : [0, 1] \rightarrow X \text{ } C^{ac}, c(0) = x, c(1) = y\}$$

for the  $F$ -distance. Note that if  $F$  is not reversible, i.e. if not  $F(\lambda v) = |\lambda|F(v)$  for all  $\lambda \in \mathbb{R}$ , we have  $d_F(x, y) \neq d_F(y, x)$  in general.

**Definition 2.3.** *A  $C^{ac}$  curve segment  $c : [a, b] \rightarrow X$  is said to be minimal, if  $l_F(c) = d_F(c(a), c(b))$ . Curves  $c : [0, \infty) \rightarrow X$ ,  $c : (-\infty, 0] \rightarrow X$ ,  $c : \mathbb{R} \rightarrow X$  are called forward rays, backward rays, minimal geodesics, respectively, if each restriction  $c|_{[a, b]}$  is a minimal segment. Set*

$$\begin{aligned} \mathcal{R}_- &:= \{v \in SX : c_v : (-\infty, 0] \rightarrow X \text{ is a backward ray}\}, \\ \mathcal{R}_+ &:= \{v \in SX : c_v : [0, \infty) \rightarrow X \text{ is a forward ray}\}, \\ \mathcal{M} &:= \{v \in SX : c_v : \mathbb{R} \rightarrow X \text{ is a minimal geodesic}\}. \end{aligned}$$

We will in this paper mainly be concerned with forward rays, the results for backward rays being completely analogous. The sets  $\mathcal{R}_-, \mathcal{R}_+$  and  $\mathcal{M}$  are  $\phi_F^t$ -invariant for  $t < 0, t > 0, t \in \mathbb{R}$ , respectively. Moreover, all sets in the above definition are closed subsets of  $SX$  and (when seen in the quotient  $SM$ ) the  $\alpha$ -,  $\omega$ -limit sets of  $\mathcal{R}_-, \mathcal{R}_+$ , respectively, are contained in  $\mathcal{M}$ .

The following lemma is a key property of rays. It excludes in particular successive intersections of rays and shows that asymptotic rays can cross only in a common initial point. The idea of the proof is classical and can be found e.g. in [Sch14a], Lemma 2.20.

**Lemma 2.4.** *Let  $v, w \in \mathcal{R}_+$  with  $\pi w = c_v(a)$  for some  $a > 0$ , but  $w \neq \dot{c}_v(a)$ . Then for all  $\delta > 0$*

$$\inf \{d_g(c_v(s), c_w(t)) : s \in [a, \infty), t \in [\delta, \infty)\} > 0.$$

**2.1. Asymptotic directions of minimal rays.** The following theorem due to H. M. Morse, which we already loosely stated as Theorem 1.1 in the introduction and call the *Morse Lemma*, is the starting point for our work. The fact that the Morse Lemma also holds in the Finsler case was first observed by E. M. Zaustinsky [Zau62]. From now on we use the fact that  $M$  carries a Riemannian metric  $g$  of strictly negative curvature.

**Theorem 2.5** (Morse Lemma [Mor24]). *There exists a constant  $D \geq 0$  depending only on  $F, g$  with the following property: For any two points  $x, y \in X$ , any  $F$ -minimal segment  $c : [0, 1] \rightarrow X$  with  $c(0) = x$ ,  $c(1) = y$  satisfies*

$$\max_{t \in [0, 1]} d_g(\gamma_{x,y}, c(t)) \leq D,$$

where  $\gamma_{x,y} \subset X$  is the unique  $g$ -geodesic segment from  $x$  to  $y$ .

As in the introduction, we write  $\mathcal{G}$  for the set of all oriented, unparametrized  $g$ -geodesics  $\gamma \subset X$ . We can think of  $\mathcal{G}$  as  $S^1 \times S^1 - \text{diag}$ , associating to  $\gamma \in \mathcal{G}$  its pair of endpoints  $(\gamma(-\infty), \gamma(\infty))$  on  $S^1$ , where

$$\gamma(\pm\infty) = \lim_{t \rightarrow \pm\infty} \gamma(t) \quad \text{in the euclidean sense in } \mathbb{C} \supset X.$$

If  $c : [0, \infty) \rightarrow X$  is a ray (with respect to  $F$ ) and  $T_n \rightarrow \infty$ , we let  $\gamma_n$  be the sequence of  $g$ -geodesic segments from  $c(0)$  to  $c(T_n)$ . Then  $\gamma_n$  converges to a unique  $g$ -ray  $\gamma : [0, \infty) \rightarrow X$  (convergence in the sense of initial velocities), independently of the choice of  $T_n$ , and we write  $c(\infty) := \gamma(\infty)$ . The uniqueness follows easily from the Morse Lemma and the fact that different  $g$ -rays initiating from  $c(0)$  diverge in  $X$  due to negative curvature of  $g$ . Moreover, if  $\gamma : \mathbb{R} \rightarrow X$  is a  $g$ -geodesic, we find a convergent subsequence of  $F$ -minimal segments from  $\gamma(-n)$  to  $\gamma(n)$ , which yields a minimal geodesic  $c : \mathbb{R} \rightarrow X$  with  $c(\pm\infty) = \gamma(\pm\infty)$ . In this way we see that the Morse Lemma holds equally well for points  $x \neq y$  in  $X \cup S^1$ . Furthermore, it is easy to show for  $v_n, v \in \mathcal{R}_\pm$ , that

$$v_n \rightarrow v \text{ in } SX \implies c_{v_n}(\pm\infty) \rightarrow c_v(\pm\infty) \text{ in } S^1.$$

**Definition 2.6.** For  $\xi \in S^1$  and  $\gamma \in \mathcal{G}$  we set

$$\begin{aligned} \mathcal{R}_\pm(\xi) &:= \{v \in \mathcal{R}_\pm : c_v(\pm\infty) = \xi\}, \\ \mathcal{M}(\gamma) &:= \{v \in \mathcal{M} : c_v(-\infty) = \gamma(-\infty) \text{ and } c_v(\infty) = \gamma(\infty)\}. \end{aligned}$$

Hence, by the discussion above, we have

$$\mathcal{R}_\pm = \bigcup \{\mathcal{R}_\pm(\xi) : \xi \in S^1\}, \quad \mathcal{M} = \bigcup \{\mathcal{M}(\gamma) : \gamma \in \mathcal{G}\}.$$

As observed already by Morse [Mor24], in each  $\mathcal{M}(\gamma)$  there are two particular minimal geodesics, which we call the *bounding geodesics* of  $\mathcal{M}(\gamma)$ . We give the proof here to indicate that it holds equally well for general Finsler metrics.



**Lemma 2.7** (bounding geodesics). *For all  $\gamma \in \mathcal{G}$ , there are two particular, non-intersecting, minimal geodesics  $c_\gamma^0, c_\gamma^1$  in  $\mathcal{M}(\gamma)$ , such that all minimal geodesics in  $\mathcal{M}(\gamma)$  lie in the strip in  $X$  bounded by  $c_\gamma^0(\mathbb{R})$  and  $c_\gamma^1(\mathbb{R})$ .*

*Moreover, if  $S \subset X$  is the closed strip between  $c_\gamma^0, c_\gamma^1$ , then any ray  $c : [0, \infty) \rightarrow X$  initiating in  $S$  with  $c(\infty) = \gamma(\infty)$  lies eventually in  $S$ , i.e. there exists  $T \geq 0$  with  $c[T, \infty) \subset S$ . The analogous statement holds for backward rays with  $c(-\infty) = \gamma(-\infty)$ .*

As a rule, we will always assume that  $c_\gamma^1$  lies left of  $c_\gamma^0$  with respect to the orientation of  $\gamma$ .

*Proof.* Let  $\gamma_n \subset X$  be a sequence of  $g$ -geodesics with  $\bar{\gamma}_n \cap \bar{\gamma} = \emptyset$  in  $X \cup S^1$  (i.e. also the endpoints on  $S^1$  are disjoint) and such that  $\gamma_n(\pm\infty) \rightarrow \gamma(\pm\infty)$ . Let us assume that  $\gamma_n$  lie left of  $\gamma$ . If  $c_n$  is a sequence of minimal geodesics in  $\mathcal{M}(\gamma_n)$ , then by minimality no  $c_n$  can intersect any minimal geodesic from  $\mathcal{M}(\gamma)$  and hence, we find a unique limit minimal geodesic  $c_\gamma^1$  in  $\mathcal{M}(\gamma)$ . Analogously one obtains  $c_\gamma^0$ . By construction,  $c_\gamma^i$  do not intersect, since their defining sequences do not intersect.

Suppose  $c : [0, \infty) \rightarrow X$  is a ray with e.g.  $c(0) \in S$  and  $c(\infty) = \gamma(\infty)$ , but  $c[T, \infty) \not\subset S$  for all  $T \geq 0$ . We then find  $t_n \rightarrow \infty$  with  $c(t_n) \notin S$ , e.g. lying left of  $c_\gamma^1$ , while  $c(t_n) \rightarrow \gamma(\infty)$ . Thus, if  $c_n \rightarrow c_\gamma^1$  as in the construction of  $c_\gamma^1$ , we obtain by the asymptotic behavior successive intersections of  $c_n$  with  $c$  for large  $n$ , contradicting minimality.  $\square$

**2.2. Horofunctions.** We introduce in this subsection a useful class of functions  $u : X \rightarrow \mathbb{R}$ , which is naturally associated to the Finsler metric  $F$ . It has been studied in various situations in the literature, in particular in the setting of Hadamard manifolds. Let us fix any “origin”  $o \in X$ . For a sequence  $x_n \in X$  with  $x_n \rightarrow \xi \in S^1$  (in the euclidean sense), any  $C_{loc}^0$  limit  $u \in C^0(X)$  of the sequence of functions

$$x \mapsto d_F(o, x_n) - d_F(x, x_n)$$

is called a *horofunction of direction*  $\xi \in S^1$ . The set of all horofunctions of direction  $\xi$  will be denoted by

$$\mathcal{H}_+(\xi) := \{u \in C^0(X) \mid \exists x_n \rightarrow \xi : u = \lim d_F(o, x_n) - d_F(\cdot, x_n)\}$$

and we set

$$\mathcal{H}_+ := \bigcup \{\mathcal{H}_+(\xi) : \xi \in S^1\}.$$

**Lemma 2.8.** *If  $x_n \in X$  with  $x_n \rightarrow \xi \in S^1$ , then the sequence of functions  $d_F(o, x_n) - d_F(\cdot, x_n) : X \rightarrow \mathbb{R}$  has  $C_{loc}^0$  limit functions  $u \in C^0(X)$ . Moreover, any so obtained function  $u \in \mathcal{H}_+(\xi)$  has the following two properties:*

- (1)  $u(y) - u(x) \leq d_F(x, y)$  for all  $x, y \in X$ ,
- (2) for all  $x \in X$  there exists a forward ray  $c : [0, \infty) \rightarrow X$  with  $c(0) = x$ ,  $c(\infty) = \xi$ ,  $F(\dot{c}) = 1$  and  $u(c(t)) - u(x) = t$  for all  $t \in [0, \infty)$ .

The proof of Lemma 2.8 can be found as Lemma 3.5 in [Sch14b].

For  $u \in \mathcal{H}_+$  we will write

$$\mathcal{J}_+(u) := \{v \in SX : u \circ c_v(t) - u \circ c_v(0) = t \ \forall t \geq 0\}.$$

It follows from property (1) in Lemma 2.8, that  $\mathcal{J}_+(u) \subset \mathcal{R}_+$ ; property (2) shows  $\pi(\mathcal{J}_+(u)) = X$ . Moreover, property (1) and the uniform equivalence of  $F$  and  $g$  show that  $u \in \mathcal{H}_+$  is Lipschitz with respect to  $d_g$ ,

$$|u(x) - u(y)| \leq c_F \cdot d_g(x, y) \quad \forall u \in \mathcal{H}_+, x, y \in X.$$

The following lemma is well-known; indications to the proof can be found in Lemma 3.2 in [Sch14b]. If  $u : X \rightarrow \mathbb{R}$  is differentiable in  $x \in X$ , write

$$\text{grad}_F u(x) := \mathcal{L}_F^{-1}(du(x)), \quad \text{where } \mathcal{L}_F(v) = \frac{1}{2}d_v F^2(v) \in T_{\pi v}^* X \subset T^* X.$$

Here  $d_v$  denotes differentiation along the fiber,  $\mathcal{L}_F : TX \rightarrow T^*X$  is the Legendre transform associated to  $F$ . Note that if  $F$  is Riemannian, then  $\text{grad}_F u$  is the usual gradient.

**Lemma 2.9.** *Let  $u \in \mathcal{H}_+$ . Then for all  $\varepsilon > 0$ ,  $u$  is differentiable in  $\pi\phi_F^\varepsilon(\mathcal{J}_+(u))$ . If  $u$  is differentiable in  $x \in X$ , then*

$$\mathcal{J}_+(u) \cap T_x X = \{\text{grad}_F u(x)\}.$$

*Moreover, the set  $\phi_F^\varepsilon(\mathcal{J}_+(u))$  for  $\varepsilon > 0$  is locally a Lipschitz graph over its projection via  $\pi$  in  $X$ .*

We have two corollaries from Lemma 2.9.

**Corollary 2.10.** *If  $u \in \mathcal{H}_+(\xi)$ , then  $\mathcal{J}_+(u) \subset \mathcal{R}_+(\xi)$ . Moreover,*

$$\mathcal{R}_+(\xi) = \bigcup \{\mathcal{J}_+(u) : u \in \mathcal{H}_+(\xi)\}.$$

*Proof.* Let  $v \in \mathcal{J}_+(u)$  and  $t > 0$ . By property (2) in Lemma 2.8, there exists a ray  $c : [0, \infty) \rightarrow X$  with  $\dot{c} \in \mathcal{J}_+(u)$  and  $c(0) = c_v(t)$ ,  $c(\infty) = \xi$ . By Lemma 2.9 we find  $\dot{c}(0) = \dot{c}_v(t)$ , i.e. also  $c_v(\infty) = c(\infty) = \xi$  and  $\mathcal{J}_+(u) \subset \mathcal{R}_+(\xi)$ . For the second claim, let  $v \in \mathcal{R}_+(\xi)$  and  $x_n := c_v(n)$ . The corresponding horofunction (called the Busemann function of  $c_v$ ) belongs to  $\mathcal{H}_+(\xi)$  and one can easily see that  $u \circ c_v(t) - u \circ c_v(0) = t$  for  $t \geq 0$  by minimality of  $c_v$ , i.e.  $v \in \mathcal{J}_+(u)$ .  $\square$

**Corollary 2.11.** *If  $u, u' \in \mathcal{H}_+$  with  $\mathcal{J}_+(u) = \mathcal{J}_+(u')$ , then  $u = u'$ .*

*Proof.* Let  $U \subset X$  be the set where both  $u, u'$  are differentiable.  $U$  has full measure by Rademacher's Theorem (horofunctions are Lipschitz). Lemma 2.9 and  $\mathcal{J}_+(u) = \mathcal{J}_+(u')$  show  $du(x) = du'(x)$  for all  $x \in U$ . Hence  $u - u'$  is constant, while  $u(o) = 0 = u'(o)$ .  $\square$

The assumption  $\dim X = 2$  can be used to find special horofunctions in  $\mathcal{H}_+(\xi)$ , as seen in the next Lemma. For its proof we refer to Appendix A of [Sch14b].

**Lemma 2.12** (bounding horofunctions). *For fixed  $\xi \in S^1$ , there exist two unique  $u_0, u_1 \in \mathcal{H}_+(\xi)$  with the following property: for all sequences  $u_n \in \mathcal{H}_+(\xi_n)$  with  $\xi_n \rightarrow \xi$  and  $\xi_n \neq \xi$ , any  $C_{loc}^0$  limit lies in  $\{u_0, u_1\}$ . More precisely, assuming the counterclockwise orientation of  $S^1$ , we have*

$$\lim_{n \rightarrow \infty} u_n = \begin{cases} u_0 & : \text{ if } \xi_n < \xi \ \forall n \\ u_1 & : \text{ if } \xi_n > \xi \ \forall n \end{cases}.$$

We will use Lemma 2.12 in order to obtain “recurrent horofunctions”. Namely, for  $\tau \in \Gamma$  and  $u \in \mathcal{H}_+(\xi)$  define

$$(\tau u)(x) := u \circ \tau^{-1}(x) - u \circ \tau^{-1}(o).$$

Then  $\tau u$  is the horofunction with

$$\mathcal{J}_+(\tau u) = \{d\tau(v) : v \in \mathcal{J}_+(u)\},$$

and in particular  $\tau u \in \mathcal{H}_+(\tau\xi)$ . Now, if  $\{\tau_k\} \subset \Gamma$  is a sequence of deck transformations, such that  $\tau_k\xi \rightarrow \xi$  and  $\xi < \tau_k\xi$  for all  $k$  in the counterclockwise orientation of  $S^1$ , then Lemma 2.12 shows

$$\tau_k u_1 \rightarrow u_1 \quad \text{in } C_{loc}^0.$$

### 3. THE PROOF OF THE MAIN THEOREM

We define the following numbers, characterizing the “width” of asymptotic directions in  $X$ .

**Definition 3.1.** *For  $\xi \in S^1$  and  $\gamma \in \mathcal{G}$  set*

$$w_{\pm}(\xi) := \sup \left\{ \liminf_{t \rightarrow \pm\infty} d_g(c_v(\mathbb{R}), c_w(t)) : v, w \in \mathcal{R}_{\pm}(\xi) \right\},$$

$$w_0(\gamma) := \sup \left\{ \inf_{t \in \mathbb{R}} d_g(c_v(\mathbb{R}), c_w(t)) : v, w \in \mathcal{M}(\gamma) \right\}.$$

**Remark 3.2.** It is easy to see that, if  $\gamma \in \mathcal{G}$  and if  $c_{\gamma}^0, c_{\gamma}^1$  are the two bounding geodesics of  $\mathcal{M}(\gamma)$  given by Lemma 2.7, then

$$w_0(\gamma) = \inf_{t \in \mathbb{R}} d_g(c_{\gamma}^0(\mathbb{R}), c_{\gamma}^1(t)).$$

In order to make use of the width, we have to connect the behavior of  $w_+$  and  $w_0$  to the group  $\Gamma$ . In the following definition we “lift” several dynamical notions of  $g$ -geodesics in the compact quotient  $M$  to the covering  $X$ , expressing them in terms of  $\mathcal{G}$  and  $\Gamma$ .

**Definition 3.3.** *A  $g$ -geodesic  $\gamma \in \mathcal{G}$  is called an axis, if there exists some  $\tau \in \Gamma - \{\text{id}\}$  with  $\tau\gamma = \gamma$ .*

*A sequence  $\{\gamma_k\} \subset \mathcal{G}$  converges to  $\gamma \in \mathcal{G}$ , if for the pairs of endpoints  $\gamma_k(-\infty) \rightarrow \gamma(-\infty)$  and  $\gamma_k(\infty) \rightarrow \gamma(\infty)$  in  $S^1$ .*

*A sequence  $\{\tau_k\} \subset \Gamma$  is positive for  $\gamma \in \mathcal{G}$ , if there exists a sequence  $\{x_k\} \subset \gamma$  with  $x_k \rightarrow \gamma(\infty) \in S^1$  (with respect to the topology of  $\mathbb{C} \supset X$ ) and a compact set  $K \subset X$ , such that  $\tau_k x_k \in K$  for all  $k$ .*

A  $g$ -geodesic  $\gamma \in \mathcal{G}$  is forward recurrent, if there exists a positive sequence  $\{\tau_k\} \subset \Gamma$  with  $\tau_k \gamma \rightarrow \gamma$ .

A subset  $G \subset \mathcal{G}$  is minimal, if it is closed under the above notion of convergence and  $\Gamma$ -invariant, i.e.  $\tau\gamma \in G$  for all  $\tau \in \Gamma$  and  $\gamma \in G$ , and if  $G$  contains no non-trivial, closed and  $\Gamma$ -invariant subsets.

A minimal set  $G \subset \mathcal{G}$  is periodic, if  $G$  consists of all  $\Gamma$ -translates of a single axis  $\gamma \in \mathcal{G}$ .

Intuitively, a sequence  $\{\tau_k\}$  is positive for  $\gamma \in \mathcal{G}$ , if  $\{\tau_k \gamma\}$  describes the behavior of  $\gamma(t)$  in the compact quotient  $M$ , as  $t \rightarrow \infty$ . Note that, if  $G_0$  is a minimal closed and  $\phi_g^t$ -invariant subset of the  $g$ -unit tangent bundle of  $M$ , then the set of  $g$ -geodesics  $G \subset \mathcal{G}$ , which project into  $G_0$ , is minimal in the sense of Definition 3.3. Also observe that, if  $G$  is minimal, then for all  $\gamma \in G$  there exists a positive (and a negative) sequence  $\{\tau_k\} \subset \Gamma$  for  $\gamma$  with  $\overline{\{\tau_k \gamma\}} = G$ . In particular, every  $\gamma \in G$  is bi-recurrent.

**Lemma 3.4** (upper semi-continuity of width). *Let  $\gamma, \gamma' \in \mathcal{G}$  and  $\{\tau_k\} \subset \Gamma$  be a positive sequence for  $\gamma$ , such that  $\tau_k \gamma \rightarrow \gamma'$ . Then  $w_+(\gamma(\infty)) \leq w_0(\gamma')$ .*

The proof of Lemma 3.4 can be found in [Sch14b], Lemma 4.5. In particular, since  $w_0(\gamma)$  is bounded for all  $\gamma \in \mathcal{G}$  by a global constant  $D(F, g)$  by the Morse Lemma, we obtain finiteness of all widths  $w_+(\xi)$  of  $\xi \in S^1$ .

**Corollary 3.5.** *If  $\gamma$  is forward recurrent, then  $w_+(\gamma(\infty)) = w_0(\gamma)$ . If  $G \subset \mathcal{G}$  is minimal, then*

$$w_\pm|_G = w_0|_G =: w(G) = \text{const.}$$

To prove Main Theorem 1.3, we need to show that

$$w_+(\xi) = 0 \quad \forall \xi \in S^1 - \text{Fix}(\Gamma).$$

Take some  $\xi \in S^1 - \text{Fix}(\Gamma)$  and any  $\gamma \in \mathcal{G}$  with  $\gamma(\infty) = \xi$ . In the compact  $g$ -unit tangent bundle of  $M$ , the  $\omega$ -limit set of  $\gamma$  contains a minimal closed and  $\phi_g^t$ -invariant set. Let  $G \subset \mathcal{G}$  be the minimal set of  $g$ -geodesics projecting into this minimal set. Lemma 3.4 and Corollary 3.5 show that

$$w_+(\xi) = w_+(\gamma(\infty)) \leq w(G),$$

so in order to prove Main Theorem 1.3, we will prove

$$(1) \quad w(G) = 0 \quad \forall \text{ minimal, non-periodic } G \subset \mathcal{G}.$$

We will have to study the way that  $g$ -geodesics  $\gamma \in G$  approximate other  $g$ -geodesics  $\gamma' \in G$ . For this, we use two lemmata, which can be found as Lemmata 4.6 and 4.7 in [Sch14b].

**Lemma 3.6.** *Let  $\gamma, \gamma' \in \mathcal{G}$  and  $\{\tau_k\} \subset \Gamma$  with  $\tau_k \gamma \rightarrow \gamma'$ , such that in  $X$  we have  $\tau_k \gamma \cap \gamma' = \emptyset \forall k$ . Then  $w_0(\gamma) = 0$ .*

**Lemma 3.7.** *Let  $\gamma, \gamma' \in \mathcal{G}$  and  $\{\tau_k\} \subset \Gamma$  with  $\tau_k \gamma \rightarrow \gamma'$ , such that  $\gamma'$  is an axis and  $\tau_k \gamma \neq \gamma' \forall k$ . Then  $w_0(\gamma) = 0$ .*

Arguing by contradiction, we assume that for some fixed minimal and non-periodic set  $G \subset \mathcal{G}$  we have

$$w(G) > 0.$$

In the following, we will use the following notion of convergence of minimal geodesics: A sequence of minimal geodesics  $c_k : \mathbb{R} \rightarrow X$  converges to a minimal geodesic  $c : \mathbb{R} \rightarrow X$ , if there exists a sequence  $\{T_k\} \subset \mathbb{R}$  with  $\dot{c}_k(T_k) \rightarrow \dot{c}(0)$ .

**Lemma 3.8.** *If there exists a non-periodic, minimal set  $G \subset \mathcal{G}$  with  $w(G) > 0$ , then there exists a bi-recurrent  $\gamma \in \mathcal{G}$ ,  $i \in \{0, 1\}$  and a backward ray  $c : (-\infty, 0] \rightarrow X$  with  $c(-\infty) = \gamma(-\infty)$ ,  $c(0) \in c_\gamma^i(\mathbb{R})$  and for  $\{j\} = \{0, 1\} - \{i\}$*

$$\liminf_{t \rightarrow -\infty} d_g(c_\gamma^j(\mathbb{R}), c(t)) = 0, \quad \liminf_{t \rightarrow -\infty} d_g(c_\gamma^i(\mathbb{R}), c(t)) = w_0(\gamma).$$

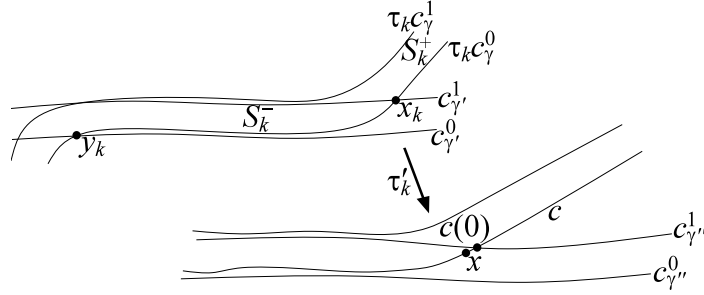


FIGURE 2. The objects in the proof of Lemma 3.8.

*Proof.* As a consequence of Lemma 3.7, the set  $G$  contains no axes. Lemma 3.6 then shows that, if  $\gamma, \gamma' \in G$  and  $\{\tau_k\} \subset \Gamma$  positive for  $\gamma$  with  $\tau_k \gamma \rightarrow \gamma'$ , then we have w.l.o.g.

$$\tau_k \gamma \cap \gamma' \neq \emptyset \quad \forall k.$$

Figure 2 depicts the following arguments. We can after passing to a subsequence assume that  $\gamma_k(\pm\infty) > \gamma'(\pm\infty)$  for all  $k$  in the counterclockwise orientation of  $S^1$ , the other case being analogous. We find a point of intersection  $x_k$  of the two minimal geodesics  $c_{\gamma'}^1(\mathbb{R}), \tau_k c_\gamma^0(\mathbb{R})$  and choose a sequence  $\{\tau'_k\} \subset \Gamma$ , such that (after passing to subsequences)  $\tau'_k \gamma' \rightarrow \gamma'' \in G$  and such that  $\{\tau'_k x_k\}$  converges to some  $x \in X$ . Let  $y_k$  be a point of intersection of  $c_{\gamma'}^0(\mathbb{R}), \tau_k c_\gamma^0(\mathbb{R})$ , then  $d_g(x_k, y_k) \rightarrow \infty$  (any limit of  $\{\tau_k c_\gamma^0\}$  lies in  $\mathcal{M}(\gamma')$  and hence left of  $c_{\gamma'}^0$ ). For  $k \rightarrow \infty$ , the sequence  $\{\tau'_k \tau_k c_\gamma^0\}$  has a convergent subsequence with a limit minimal geodesic  $c : \mathbb{R} \rightarrow X$ , now connecting  $\lim_k \tau'_k y_k = \gamma''(-\infty)$  to  $\lim_k \tau'_k x_k = x$ .

We claim that the minimal geodesic  $c(t)$  has to leave the strip between  $c_{\gamma''}^i$  to the left, as  $t \rightarrow +\infty$ . To see this, observe that at each  $x_k$ , left of  $c_{\gamma'}^1$ , there is a strip  $S_k^+$  of width  $w(G)$  in the positive end of the strip between

$\tau_k c_\gamma^i$ , which will under  $\{\tau_k'\}$  converge to a strip of width  $w(G)$  left of  $c_{\gamma''}^1$  starting at  $x$ . Since  $w_+(\gamma''(\infty)) = w(G) < 2w(G)$  by  $w(G) > 0$ , the limit of the strips  $\tau_k' S_k^+$  cannot have asymptotic direction  $\gamma''(\infty)$ .

Hence, we can assume that  $c(0) \in c_{\gamma''}^1(\mathbb{R})$ . Moreover, the strips  $S_k^-$  between  $\tau_k c_\gamma^i$  left of  $x_k$  will, under application of  $\{\tau_k'\}$ , converge to a strip in the negative end of the strip between  $c_{\gamma''}^i$  (by  $\tau_k' y_k \rightarrow \gamma''(-\infty)$  it has to lie in bounded distance of  $\gamma''$  and by the same reasoning as above it cannot lie outside of the strip between  $c_{\gamma''}^i$ ). Hence, we have

$$\liminf_{t \rightarrow -\infty} d_g(c_{\gamma''}^0(\mathbb{R}), c(t)) = 0, \quad \liminf_{t \rightarrow -\infty} d_g(c_{\gamma''}^1(\mathbb{R}), c(t)) = w(G) = w_0(\gamma'').$$

The  $g$ -geodesic  $\gamma'' \in G$  is the  $\gamma$  from the statement of the lemma. Bi-recurrence of  $\gamma''$  follows from the minimality of  $G$ .  $\square$

For simplicity, we turn the picture around, replacing negative recurrence etc. by forward recurrence etc. (formally by considering the Finsler metric  $v \mapsto F(-v)$ ).

**Definition 3.9.** *Two rays  $c_0, c_1 : [0, \infty) \rightarrow X$  with  $c_i(\infty) = \xi$  are fully separated at  $+\infty$ , if*

$$\liminf_{t \rightarrow \infty} d_g(c_0[0, \infty), c_1(t)) = w_+(\xi).$$

*Analogously we explain when two rays in  $\mathcal{R}_-(\gamma(-\infty))$  is fully separated from each other at  $-\infty$ .*

In view of Lemma 3.8, the proof of the following theorem will finish the proof of Main Theorem 1.3.

**Theorem 3.10.** *If  $\gamma \in \mathcal{G}$  is bi-recurrent with  $w_0(\gamma) > 0$  and if  $c_*$  is any of the bounding geodesics  $c_\gamma^0, c_\gamma^1$  of  $\mathcal{M}(\gamma)$ , then there cannot exist a ray  $c$  in  $\mathcal{R}_+(\gamma(\infty))$  initiating from  $c_*(\mathbb{R})$  and being fully separated from  $c_*$  at  $+\infty$ .*

We fix in the following the bi-recurrent  $g$ -geodesic  $\gamma \in \mathcal{G}$  and a positive sequence  $\{\tau_k\} \subset \Gamma$  for  $\gamma$  with  $\tau_k \gamma \rightarrow \gamma$ . Let us moreover assume (using Corollary 3.5)

$$w_0(\gamma) = w_+(\gamma(\infty)) = w_-(\gamma(-\infty)) > 0.$$

We prove three lemmata before turning to the proof of Theorem 3.10.

**Lemma 3.11.** *Let  $\xi \in S^1$  and  $u \in \mathcal{H}_+(\xi)$ . Then for all  $v \in \mathcal{R}_+(\xi)$  the function*

$$t \in [0, \infty) \mapsto t - u \circ c_v(t)$$

*is bounded and monotonically increasing.*

*Proof.* We let  $c, c_0 : [0, \infty) \rightarrow X$  be two rays with  $\dot{c}, \dot{c}_0 \in \mathcal{R}_+(\xi)$ . Recall that by property (1) in Lemma 2.8 and  $u \in \mathcal{H}_+$ , we have for  $a \leq b$

$$a - u \circ c(a) \leq b - u \circ c(b),$$

hence the lower bound by  $-u \circ c(0)$  and monotonicity are trivial. By the Morse Lemma, we find

$$B := \sup_{t \geq 0} d_g(c_0[0, \infty), c(t)) < \infty,$$

and hence for all  $t \geq 0$  we find some  $s \in \mathbb{R}$  with  $d_g(c_0(s), c(t)) \leq B$ . We first assume that  $s \leq t$ , then by minimality of  $c, c_0$  and the triangle inequality

$$\begin{aligned} t &= d_F(c(0), c(t)) \\ &\leq d_F(c(0), c_0(0)) + d_F(c_0(0), c_0(s)) + d_F(c_0(s), c(t)) \\ &\leq d_F(c(0), c_0(0)) + s + c_F \cdot B. \end{aligned}$$

If  $t \leq s$ , then similarly

$$\begin{aligned} s &= d_F(c_0(0), c_0(s)) \\ &\leq d_F(c_0(0), c(0)) + d_F(c(0), c(t)) + d_F(c(t), c_0(s)) \\ &\leq d_F(c_0(0), c(0)) + t + c_F \cdot B. \end{aligned}$$

In any case,  $|t - s|$  is bounded from above by  $d_F(c(0), c_0(0)) + c_F B$  and hence

$$\begin{aligned} \frac{1}{c_F} \cdot d_g(c_0(t), c(t)) &\leq d_F(c_0(t), c(t)) \\ &\leq d_F(c_0(t), c_0(s)) + d_F(c_0(s), c(t)) \\ &\leq |t - s| + c_F \cdot B \\ &\leq d_F(c(0), c_0(0)) + 2 \cdot c_F \cdot B =: C. \end{aligned}$$

The fact that  $u$  is Lipschitz with respect to  $d_g$  with Lipschitz constant  $c_F$  shows

$$|u \circ c(t) - u \circ c_0(t)| \leq c_F \cdot d_g(c_0(t), c(t)) \leq c_F^2 \cdot C.$$

Assume now that  $\dot{c}_0 \in \mathcal{J}_+(u)$ , then by definition  $u \circ c_0(t) - t \equiv u \circ c_0(0)$  and hence

$$|u \circ c(t) - t| \leq |u \circ c(t) - u \circ c_0(t)| + |u \circ c_0(t) - t| \leq c_F^2 \cdot C + |u \circ c_0(0)|.$$

□

The next lemma shows that, given the “homotopy information”  $\{\tau_k\}$ , that leads to forward recurrence of  $\gamma$ , we can also find forward recurrent motions in  $\mathcal{M}(\gamma)$  with the same “homotopy information”.

**Lemma 3.12.** *There exists a pair of minimal geodesics  $c_0, c_1$  in  $\mathcal{M}(\gamma)$  and sequences  $T_k^0, T_k^1 \rightarrow \infty$ , such that  $\dot{c}_0(0)$  is a limit point of  $\{d\tau_k(\dot{c}_0(T_k^0))\}_k$  and  $\dot{c}_1(0)$  is a limit point of  $\{d\tau_k(\dot{c}_1(T_k^1))\}_k$ , while  $c_0$  and  $c_1$  are fully separated at  $-\infty$  and at  $+\infty$ .*

Note that it will now be enough to prove Theorem 3.10 for  $c_* \in \{c_0, c_1\}$  given by Lemma 3.12, since any ray initiating from one of the bounding geodesics will cross the corresponding  $c_i$  by full separation.

*Proof.* Let us say that a subset  $A \subset \mathcal{M}(\gamma)$  is  $\{\tau_k\}$ -invariant, if for all  $v$  in  $A$  any limit geodesic of  $\{\tau_k c_v\}$  belongs again to  $A$ . It follows from  $w_0(\gamma) > 0$ , that the following two subsets are  $\{\tau_k\}$ -invariant:

$$A_0 := \{v \in \mathcal{M}(\gamma) : \inf_{t \in \mathbb{R}} d_g(c_\gamma^1(\mathbb{R}), c_v(t)) \geq w_0(\gamma)\},$$

$$A_1 := \{v \in \mathcal{M}(\gamma) : \inf_{t \in \mathbb{R}} d_g(c_\gamma^0(\mathbb{R}), c_v(t)) \geq w_0(\gamma)\}.$$

Obviously, both sets  $A_0, A_1$  are closed sets and any pair of minimal geodesics  $c_0 \in A_0, c_1 \in A_1$  is fully separated at  $\pm\infty$ . Moreover, the bounding geodesics  $c_\gamma^i$  lie in  $A_i$ , i.e.  $A_i \neq \emptyset$ . By full separation, one can also easily show that both sets  $A_i$  define a lamination of  $X$  (any two geodesics from  $A_0$ , say, come close at  $\pm\infty$ , then use Lemma 2.4).

We will argue for the set  $A_0$ , the case of  $A_1$  being the same. We wish to find a “minimal subset”  $\hat{A}_0$  of  $A_0$ . For this, let  $\mathfrak{A}_0$  be the collection of non-empty, closed,  $\{\tau_k\}$ -invariant subsets of  $A_0$ , then  $\mathfrak{A}_0 \neq \emptyset$  by  $A_0 \in \mathfrak{A}_0$  and  $\mathfrak{A}_0$  is partially ordered by  $\subset$ . Moreover, any linearly ordered subset of  $\mathfrak{A}_0$  has the intersection of its members as a  $\subset$ -smallest element (the intersection is non-empty by Cantor’s Intersection Theorem, if we think of the lamination  $A_0$  as a set of points in some compact, transverse interval). Hence, Zorn’s Lemma shows the existence of the desired  $\hat{A}_0$ , which has thus the property of not containing any non-trivial, closed,  $\{\tau_k\}$ -invariant subsets.

Since  $A_0$  defines a lamination of  $X$  and  $\hat{A}_0$  is closed, there exists an uppermost minimal geodesic  $c_0$  in  $\hat{A}_0$ , and we claim that  $c_0$  is  $\{\tau_k\}$ -recurrent. Namely, let  $\hat{c}_0$  be the uppermost limit geodesic of  $\{\tau_k c_0\}$ , then  $\hat{c}_0$  belongs to  $\hat{A}_0$ , so does not lie above  $c_0$ . But by minimality of  $\hat{A}_0$ , the subset of  $\hat{A}_0$  consisting of all the  $\{\tau_k\}$ -limit geodesics of  $\hat{A}_0$  equals  $\hat{A}_0$  and hence  $c_0$  is the limit of some  $c$  in  $\hat{A}_0$ . By the preservation of the ordering in the foliation in  $A_0$  under  $\{\tau_k\}$ -limits,  $c_0$  is also the limit of itself: the limiting process of  $\tau_k c \rightarrow c_0$  pushes also the limiting process of  $\tau_k c_0 \rightarrow \hat{c}_0$  up to  $c_0$ .  $\square$

Let  $c_* : \mathbb{R} \rightarrow X$  be one of the minimal geodesics and  $\{T_k^*\}$  be the corresponding sequence given by Lemma 3.12. We replace  $\{\tau_k\}$  by a subsequence in order to obtain  $d\tau_k(\dot{c}_*(T_k^*)) \rightarrow \dot{c}_*(0)$ . Moreover, we let  $c : [0, \infty) \rightarrow X$  be a ray with  $c(\infty) = c_*(\infty)$ , such that  $c, c_*$  are fully separated at  $+\infty$ . The idea behind the following lemma is that  $c$  has limits, which have the same average displacement along the “homotopy path”  $\{\tau_k\}$  as  $c_*$ .

**Lemma 3.13.** *With the above notation, there exists a sequence  $S_k \rightarrow \infty$ , such that*

$$\liminf_{k \rightarrow \infty} d_g(c(S_k), \tau_k c(T_k^* + S_k)) = 0.$$

*Proof.* Let  $c_1$  in  $\mathcal{M}(\gamma)$  be a limit geodesic of  $\{\tau_k c\}$  and after taking a further subsequence, we may assume  $\tau_k c \rightarrow c_1$ . Since  $c$  is fully separated from  $c_*$ , so will be  $c_1$ , such that

$$\liminf_{S \rightarrow \infty} d_g(c_1(\mathbb{R}), c(S)) = 0.$$



We fix  $\varepsilon > 0$  and let  $S \geq 0$ , such that  $d_g(c_1(\mathbb{R}), c(S)) \leq \varepsilon$ . Since  $\tau_k c \rightarrow c_1$ , we then find some  $k_0 \in \mathbb{N}$ , such that (using positivity of  $\{\tau_k\}$  for  $c_*$ )

$$(2) \quad \exists T_k \rightarrow \infty : \quad d_g(c(S), \tau_k c(T_k + S)) \leq 2\varepsilon \quad \forall k \geq k_0.$$

Moreover, by Lemma 3.11, we may also assume that  $S$  is large enough, such that

$$|[(S + T) - u \circ c(S + T)] - [S - u \circ c(S)]| \leq \varepsilon \quad \forall T \geq 0.$$

and hence

$$(3) \quad |T_k - u \circ c(S + T_k) + u \circ c(S)| \leq \varepsilon.$$

We let  $u = \lim \tau_k u \in \mathcal{H}_+(c_*(\infty))$  (cf. the discussion after Lemma 2.12 for the existence of such  $u$ ). Let us see that

$$(4) \quad T = u \circ c_*(T) - u \circ c_*(0) \quad \forall T \geq 0.$$

Indeed, by  $\tau_k u \rightarrow u$  and  $\tau_k c_*(\cdot + T_k^*) \rightarrow c_*$  (both in  $C_{loc}^0$ ) we find

$$\begin{aligned} T - u \circ c_*|_0^T &= \lim_{k \rightarrow \infty} T - (\tau_k u) \circ \tau_k c_*(\cdot + T_k^*)|_0^T \\ &= \lim_{k \rightarrow \infty} T - u \circ \tau_k^{-1} \circ \tau_k c_*(\cdot + T_k^*)|_0^T \\ &= \lim_{k \rightarrow \infty} T - u \circ c_*(\cdot + T_k^*)|_0^T \\ &= 0 \end{aligned}$$

by Lemma 3.11. Now observe, that for  $k$  sufficiently large we have

$$\begin{aligned} &|T_k - T_k^*| \\ &\leq |T_k - u \circ c(S + T_k) + u \circ c(S)| + |u \circ c(S + T_k) - u \circ c(S) - T_k^*| \\ &\stackrel{(3),(4)}{\leq} \varepsilon + |u \circ c(S + T_k) - u \circ c(S) - u \circ c_*(T_k^*) + u \circ c_*(0)| \\ &\leq \varepsilon + |u \circ c(S + T_k) - u \circ \tau_k^{-1} c(S)| \\ &\quad + |u \circ \tau_k^{-1} c(S) - u \circ c(S) - u \circ \tau_k^{-1} c_*(0) + u \circ c_*(0)| \\ &\quad + |u \circ \tau_k^{-1} c_*(0) - u \circ c_*(T_k^*)| \\ &\leq \varepsilon + c_F \cdot d_g(c(S + T_k), \tau_k^{-1} c(S)) \\ &\quad + |(\tau_k u) \circ c(S) - u \circ c(S) - (\tau_k u) \circ c_*(0) + u \circ c_*(0)| \\ &\quad + c_F \cdot d_g(\tau_k^{-1} c_*(0), c_*(T_k^*)) \\ &\stackrel{(2)}{\leq} \varepsilon + c_F \cdot 2\varepsilon \\ &\quad + |(\tau_k u) \circ c(S) - u \circ c(S)| + |(\tau_k u) \circ c_*(0) - u \circ c_*(0)| \\ &\quad + c_F \cdot d_g(c_*(0), \tau_k c_*(T_k^*)). \end{aligned}$$

By  $\tau_k u \rightarrow u$  and  $\tau_k c_*(\cdot + T_k^*) \rightarrow c_*$ , we find for large  $k$ , that

$$|T_k - T_k^*| \leq 2\varepsilon(1 + c_F),$$

so that (2) and the triangle inequality show

$$\begin{aligned}
& d_g(c(S), \tau_k c(T_k^* + S)) \\
& \leq d_g(c(S), \tau_k c(T_k + S)) + d_g(\tau_k c(T_k + S), \tau_k c(T_k^* + S)) \\
& \leq 2\varepsilon + |T_k - T_k^*| \\
& \leq 2\varepsilon + 2\varepsilon(1 + c_F)
\end{aligned}$$

for  $k$  sufficiently large and  $S$  as chosen above.  $\square$

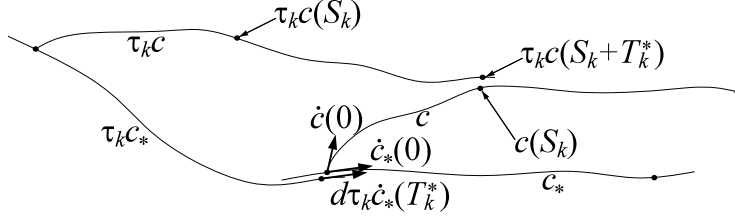


FIGURE 3. The argument in the Proof of Theorem 3.10.

*Proof of Theorem 3.10.* Let  $c_*$  and  $\{T_k^*\}$  be given by Lemma 3.12, such that after taking subsequences we have  $d\tau_k(\dot{c}(T_k^*)) \rightarrow \dot{c}_*(0)$ . Moreover, let as above  $c : [0, \infty) \rightarrow X$  be a ray with  $c(0) = c_*(0)$  and  $c(\infty) = c_*(\infty)$ . Assuming  $c$  to be fully separated from  $c_*$  at  $+\infty$ , in particular  $\dot{c}(0) \neq \dot{c}_*(0)$ , we have to arrive at a contradiction. We saw in Lemma 3.13, that we find a sequence  $S_k \rightarrow \infty$  with  $\liminf_{k \rightarrow \infty} d_g(c(S_k), \tau_k c(T_k^* + S_k)) = 0$  and after passing to another subsequence, we suppose

$$(5) \quad d_F(c(S_k), \tau_k c(T_k^* + S_k)) \rightarrow 0, \quad k \rightarrow \infty.$$

By  $\dot{c}(0) \neq \dot{c}_*(0)$  we find a small  $\delta > 0$  and, depending on the angle between  $\dot{c}_*(0)$  and  $\dot{c}(0)$ , some  $\varepsilon > 0$  with

$$d_F(c_*(-\delta), c(\delta)) \leq 2\delta - \varepsilon.$$

By  $d\tau_k(\dot{c}_*(T_k^*)) \rightarrow \dot{c}_*(0)$  we then obtain for large  $k$ , that

$$(6) \quad d_F(\tau_k c_*(T_k^* - \delta), c(\delta)) \leq 2\delta - \varepsilon/2.$$

We find by minimality of  $\tau_k c$  and  $c_*(0) = c(0)$ , that for sufficiently large  $k$

$$\begin{aligned}
T_k^* + S_k &= d_F(\tau_k c(0), \tau_k c(S_k + T_k^*)) \\
&\leq d_F(\tau_k c_*(0), \tau_k c_*(T_k^* - \delta)) + d_F(\tau_k c_*(T_k^* - \delta), c(\delta)) \\
&\quad + d_F(c(\delta), c(S_k)) + d_F(c(S_k), \tau_k c(S_k + T_k^*)) \\
&\stackrel{(5),(6)}{\leq} d_F(\tau_k c_*(0), \tau_k c_*(T_k^* - \delta)) + 2\delta - \varepsilon/2 \\
&\quad + d_F(c(\delta), c(S_k)) + \varepsilon/4 \\
&= T_k^* + S_k - \varepsilon/4.
\end{aligned}$$

This is a contradiction.  $\square$

### 3.1. The proofs of the corollaries.

*Proof of Corollary 1.5.* The fact that rays with common endpoint  $\xi \notin \text{Fix}(\Gamma)$  can intersect only in a common initial point can be deduced directly from  $w_+(\xi) = 0$  and Lemma 2.4.  $\square$

*Proof of Corollary 1.7.* Use the arguments in the proof of Lemma 4.5 in [GKOS14].  $\square$

*Proof of Corollary 1.8.* Let  $\xi \notin \text{Fix}(\Gamma)$  and  $u \in \mathcal{H}_+(\xi)$ . If  $v \in \mathcal{R}_+(\xi)$  and  $\varepsilon > 0$ , then by Corollary 1.5, the ray  $c_v : [\varepsilon, \infty) \rightarrow X$  belongs to  $\mathcal{J}_+(u)$ . The closedness of  $\mathcal{J}_+(u)$  shows  $v \in \mathcal{J}_+(u)$ , i.e.  $\mathcal{R}_+(\xi) \subset \mathcal{J}_+(u)$  and using Corollary 2.10 we have  $\mathcal{R}_+(\xi) = \mathcal{J}_+(u)$  for all  $u \in \mathcal{H}_+(\xi)$ . Hence Corollary 2.11 shows the uniqueness of the horofunction  $u \in \mathcal{H}_+(\xi)$ .  $\square$

*Proof of Corollary 1.9.* The fact that  $\phi_F^\varepsilon \mathcal{R}_+(\xi)$  with  $\xi \notin \text{Fix}(\Gamma)$  and  $\varepsilon > 0$  is locally a Lipschitz graph follows directly from Lemma 2.9 and  $\mathcal{R}_+(\xi) = \mathcal{J}_+(u)$  in Corollary 1.8.  $\square$

*Proof of Corollary 1.10.* Let  $\xi \notin \text{Fix}(\Gamma)$  and, using the bounding geodesics of  $\mathcal{M}(\gamma)$  from Lemma 2.7, set

$$L := \bigcup \{c_\gamma^i(\mathbb{R}) : \gamma \in \mathcal{G}, \gamma(\infty) = \xi, i = 0, 1\},$$

$$A := \{C \subset X : C \text{ connected component of } X - L\}.$$

$L$  defines a closed lamination of  $X$  by Corollary 1.5 (for closedness recall the construction of the  $c_\gamma^i$ ) and, moreover, for any  $\gamma \in \mathcal{G}$  with  $\gamma(\infty) = \xi$  and  $c_\gamma^0(\mathbb{R}) \neq c_\gamma^1(\mathbb{R})$ , the (connected) strip  $C_\gamma \subset X - L$  between the  $c_\gamma^i(\mathbb{R})$  is an element of  $A$ . The claim follows, since  $A$  is countable:  $X$  is the union of countably many compact sets  $K_n$ , and each  $K_n$  can contain at most countably many disjoint open sets.

For  $\xi \in \text{Fix}(\Gamma)$ , the set  $L$  above also defines a lamination of  $X$ , which can be seen directly from Theorem 1.2 and Lemma 2.4. Hence, the above reasoning applies to all  $\xi \in S^1$ .  $\square$

*Proof of Corollary 1.11.* Fixing  $\xi \in S^1$ , consider the lamination  $L$  of  $X$  as in the proof of Corollary 1.10. If  $x \in L$ , then  $x$  lies on some  $c_\gamma^i$ , which determines  $\gamma$  uniquely. If  $x \in X - L$ , then  $x$  lies in some open strip  $C_\gamma$ , i.e. between  $c_\gamma^0$  and  $c_\gamma^1$ , again determining  $\gamma$  uniquely.  $\square$

*Proof of Corollary 1.13.* Let  $\xi \in \text{Fix}(\Gamma)$ . Observe that, if  $\mathcal{M}_{per}(\gamma)$  contains only one minimal geodesic, then by Theorem 1.2, all rays in  $\mathcal{R}_+(\xi)$  are in fact asymptotic. Hence, Main Theorem 1.3 holds for  $\xi \in \text{Fix}(\Gamma)$  and so do its corollaries.  $\square$

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